

# Partial wave expansion for photoproduction of two pseudoscalars on a nucleon

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(Dated: January 30, 2012)

The amplitudes for photoproduction of two pseudoscalars on a nucleon are expanded in the overall c.m. frame in a model independent way with respect to the contribution of the final state partial wave of total angular momentum  $J$  and its projection on the normal to the plane spanned by the momenta of the final particles. The expansion coefficients which are analogues to the multipole amplitudes for single meson photoproduction contain the complete information about the reaction dynamics. Results of an explicit evaluation are presented for the moments  $W_{jm}$  of the inclusive angular distribution of an incident photon beam with respect to the c.m. coordinate system defined by the final particles taking photoproduction of  $\pi^0\pi^0$  and  $\pi^0\eta$  as an example.

PACS numbers: 13.60.Le, 13.75.-n, 21.45.+v, 25.20.Lj

## I. INTRODUCTION

The study of multiple meson production is essential for understanding the properties of baryonic resonances, especially of those having sizeable inelasticities and for which only a weak evidence from elastic  $\pi N$  scattering exists. According to the quark model calculation of [1], at least below 2 GeV some of these resonances must be strongly coupled to  $\pi\pi N$  and  $\pi\eta N$  channels. Therefore, present experiments on  $\pi\pi$  and  $\pi\eta$  photoproduction have become a center of attention in programs discussed at various research centers, and a number of new accurate data have already been reported [2–9].

Improvements in the quality of the data have made it possible to perform rather detailed theoretical analyses of photoproduction of two pseudoscalars. The  $\pi\pi$  as well as  $\pi\eta$  models have already been the object of several studies [10–18]. Mainly they cover the second and third resonance regions describing with varying degrees of success the existing data and predicting the results of new measurements. A typical analysis is based on an isobar model approach. Its key assumption is that the amplitude is a coherent sum of background and resonances usually parametrized in terms of effective Lagrangeans. As a rule, the resonance part contains  $s$ -channel resonances decaying into  $\pi\pi N$  or  $\pi\eta N$  via intermediate formation of meson-nucleon and meson-meson isobars. As adjustable parameters one usually takes the masses and partial decay widths of the resonances as well as their electromagnetic coupling constants.

Within this method, angular momentum decomposition of the amplitude is ruled by partial wave transitions of the resonance states to quasi-two-body states, like  $\pi\Delta$  or  $\eta\Delta$ . Clearly, such an approach can not be viewed as a general partial wave analysis, since it crucially depends on the assumptions about the production mechanism. Therefore, it results in various uncertainties, primarily in the non-uniqueness of existing solutions, since the same observables may equally well be described with different sets of parameters. Consequently, in spite of a general qualitative agreement among the models, significant quantitative discrepancies still remain. Additional limitations may arise from inadequacies of the isobar model description, such as violation of unitarity, nonrelativistic dynamics *etc.*, whose impact on the description of the processes under discussion remains unknown. For instance, unitarity conditions may be important in the region where many production channels are open.

It is worth to note that one of the main reasons for the lack of a rigorous partial wave analysis for  $\pi\pi$  and  $\pi\eta$  photoproduction is that there is no general recipe to deal with reactions involving three particles in the final state. In contrast to single meson photoproduction one faces here the technical problems associated with three-body kinematics, where the particle energies and angles are distributed continuously. As a consequence, a conventional partial wave decomposition of the final state does not provide a multipole representation for practical applications, primarily since there exists a variety of ways to successively couple angular momenta of the participating particles to a total angular momentum.

In this paper we present an alternative method by using a partial wave expansion for the photon induced production of two pseudoscalars on a nucleon, which should be of minimal model dependence. It is based on the correct determination of the partial wave amplitudes for these reactions with no built-in prejudices concerning the production mechanism. Similar method have been used to analyse pion production in  $\pi N$  collisions, see, for example Refs. [19, 20].

The paper is organized as follows. In the next section we introduce the partial wave expansion and construct the transition amplitude for photoproduction of two pseudoscalar mesons. In Sect. III we use the so far developed formalism to discuss some gross features of  $\pi^0\pi^0$  and  $\pi^0\eta$  photoproduction. Finally, some general conclusions are

drawn in Sect. IV.

## II. THE FORMALISM

In this section, we collect the formulas used in the present analysis. As a starting point the formal results of Ref. [21] are used. There the formal expressions for the helicity amplitudes as well as for the cross section and the recoil polarization were derived, including various polarization asymmetries with respect to polarized photons and nucleons.

### A. The T matrix

We consider here the photoproduction of two pseudoscalar mesons, denoted  $m_1$  and  $m_2$  with masses  $M_1$  and  $M_2$ , respectively. Firstly we determine the  $T$ -matrix elements of the electromagnetic  $m_1 m_2$  production current  $\vec{J}_{\gamma m_1 m_2}$  between the initial nucleon and the final  $m_1 m_2 N$  state. The four-momenta of incoming photon, outgoing mesons, initial and final nucleons are denoted by  $(\omega_\gamma, \vec{k})$ ,  $(\omega_1, \vec{q}_1)$ ,  $(\omega_2, \vec{q}_2)$ ,  $(E_i, \vec{p}_i)$ , and  $(E, \vec{p})$ , respectively. The helicities of photon and initial and final nucleons are denoted by  $\lambda$ ,  $\mu$ , and  $\nu$ , respectively. In a general frame the transition matrix element is given by

$$T_{\nu\lambda\mu} = -^{(-)}\langle \vec{p}, \vec{q}, \nu | \vec{\varepsilon}_\lambda \cdot \vec{J}_{\gamma m_1 m_2}(0) | \vec{p}_i, \mu \rangle, \quad (1)$$

where for the description of the final state we choose the final nucleon momentum  $\vec{p} = (p, \theta_p, \phi_p)$  and the relative momentum of the two mesons  $\vec{q} = \frac{1}{2}(\vec{q}_1 - \vec{q}_2) = (q, \theta_q, \phi_q)$ . For the following formal considerations the knowledge of the specific form of the current  $\vec{J}_{\gamma m_1 m_2}$  is not needed.

After separation of the overall c.m.-motion the general form of the  $T$ -matrix is given by

$$T_{\nu\lambda\mu} = -^{(-)}\langle \vec{p}, \vec{q}, \nu | J_{\gamma m_1 m_2, \lambda}(\vec{k}) | \mu \rangle. \quad (2)$$

It is convenient to introduce a partial wave decomposition of the outgoing final state according to

$$\begin{aligned} ^{(-)}\langle \vec{q}, \vec{p}, \nu | &= \frac{1}{4\pi} \sum_{l_p j_p m_p l_q m_q J M} \hat{l}_p \hat{l}_q (l_p 0 \frac{1}{2} \nu | j_p \nu) (j_p m_p l_q m_q | J M) D_{\nu m_p}^{j_p}(\phi_p, -\theta_p, -\phi_p) \\ &\times D_{0 m_q}^{l_q}(\phi_q, -\theta_q, -\phi_q) ^{(-)}\langle q p; ((l_p \frac{1}{2}) j_p l_q) J M |, \end{aligned} \quad (3)$$

where the “hat” symbol means, for example,  $\hat{l}_q = \sqrt{2l_q + 1}$ . Furthermore,  $l_q$  and  $m_q$  denote total angular momentum and projection, respectively, of the two mesons,  $l_p$ ,  $j_p$ , and  $m_p$  orbital and total nucleon angular momentum and its projection, respectively, and  $J$  and  $M$  the total angular momentum of the partial wave and its projection. All projections refer to a quantization axes to be determined later. For the rotation matrices  $D_{m'm}^j$  we follow the convention of Rose [22].

The multipole decomposition of the current reads with  $\vec{k} = (k, \theta_\gamma, \phi_\gamma)$

$$J_{\gamma m_1 m_2, \lambda}(\vec{k}) = -\sqrt{2\pi} \sum_{L M_L} i^L \hat{L} \mathcal{O}_{M_L}^{\lambda L}(k) D_{M_L \lambda}^L(\phi_\gamma, \theta_\gamma, -\phi_\gamma), \quad (4)$$

where  $\mathcal{O}_{M_L}^{\lambda L}$  contains the transverse electric and magnetic multipoles

$$\mathcal{O}_{M_L}^{\lambda L} = E_{M_L}^L + \lambda M_{M_L}^L. \quad (5)$$

For the initial nucleon state we have

$$|\frac{1}{2}\mu\rangle = (-1)^{\frac{1}{2}+\mu} \sum_{m=\pm 1/2} |\frac{1}{2}m\rangle D_{m-\mu}^{1/2}(\phi_\gamma, \theta_\gamma, -\phi_\gamma). \quad (6)$$

Using the Wigner-Eckart theorem and the sum rule for rotation matrices

$$\sum_{M_L m} \begin{pmatrix} J & L & \frac{1}{2} \\ -M & M_L & m \end{pmatrix} D_{m-\mu}^{1/2}(R) D_{M_L \lambda}^L(R) = (-1)^{\lambda-\mu-M} \begin{pmatrix} J & L & \frac{1}{2} \\ \mu-\lambda & \lambda & -\mu \end{pmatrix} D_{M \lambda-\mu}^J(R), \quad (7)$$

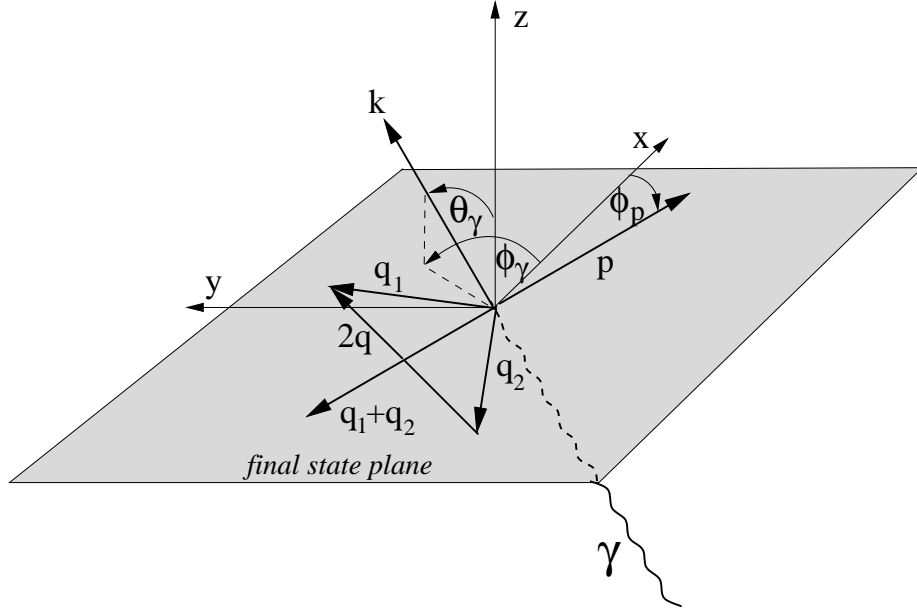


FIG. 1: Definition of the coordinate system in the c.m. system.

one obtains

$$\begin{aligned}
 T_{\nu\lambda\mu} = & \frac{(-1)^{\nu+\lambda}}{2\sqrt{2\pi}} \sum_{Ll_p j_p m_p l_q m_q J M} (-1)^{l_p+j_p+l_q+J-M} i^L \widehat{L} \widehat{J} \widehat{l}_q \widehat{l}_p \widehat{j}_p \begin{pmatrix} l_p & \frac{1}{2} & j_p \\ 0 & \nu & -\nu \end{pmatrix} \\
 & \times \begin{pmatrix} j_p & l_q & J \\ m_p & m_q & -M \end{pmatrix} \begin{pmatrix} J & L & \frac{1}{2} \\ \mu-\lambda & \lambda & -\mu \end{pmatrix} \langle p q; ((l_p \frac{1}{2}) j_p l_q) J || \mathcal{O}^{\lambda L} || \frac{1}{2} \rangle \\
 & \times D_{\nu m_p}^{j_p}(\phi_p, -\theta_p, -\phi_p) D_{0 m_q}^{l_q}(\phi_q, -\theta_q, -\phi_q) D_{M \lambda-\mu}^J(\phi_\gamma, \theta_\gamma, -\phi_\gamma). \quad (8)
 \end{aligned}$$

Parity conservation results in the following symmetry relation

$$T_{-\nu-\lambda-\mu}(\Omega_q, \Omega_p, \Omega_\gamma) = (-1)^{\lambda-\mu-\nu} T_{\nu\lambda\mu}(\bar{\Omega}_q, \bar{\Omega}_p, \bar{\Omega}_\gamma), \quad (9)$$

where for  $\Omega = (\theta, \phi)$  we have introduced the notation  $\bar{\Omega} = (\theta, -\phi)$ .

Now we turn to the choice of our coordinate system in the overall center-of-momentum frame. We use the so-called "rigid body" system  $K_{fs}$ , associated with the final state plane spanned by the final three particles, in which the  $z$ -axis is taken to be the normal to this plane and parallel to  $\vec{p} \times \vec{q}_1$ . Thus the  $x$ - and  $y$ -axes are in the final scattering plane (see Fig. 1).

At a given three-particle invariant energy  $W$ , the relative orientation of the final particles within the final state plane is characterized by three independent variables for which we take the angle  $\phi_p$  of the final nucleon momentum and the energies of the two mesons,  $\omega_1$  and  $\omega_2$  (see Fig. 1). After straightforward algebra one obtains for the final nucleon momentum  $p$

$$p = |\vec{p}| = \sqrt{(W - \omega_1 - \omega_2)^2 - M_N^2}, \quad (10)$$

and for the relative momentum  $q$  of the two mesons

$$q^2 = \frac{1}{2}(\omega_1^2 + \omega_2^2 - M_1^2 - M_2^2) - \frac{p^2}{4}. \quad (11)$$

The orientation of the chosen coordinate system with respect to the beam axes may be specified by  $\Omega_\gamma = (\phi_\gamma, \theta_\gamma)$ , the spherical angles of the photon momentum  $\vec{k}$  with respect to  $K_{fs}$ . One readily notes that in this coordinate system one has  $\theta_p = \theta_q = \pi/2$  and therefore

$$D_{\nu m_p}^{j_p}(\phi_p, -\theta_p, -\phi_p) = (-1)^{\nu-m_p} d_{\nu m_p}^{j_p}(\pi/2) e^{-i(\nu-m_p)\phi_p}, \quad (12)$$

$$D_{0 m_q}^{l_q}(\phi_q, -\theta_q, -\phi_q) = (-1)^{m_q} d_{0 m_q}^{l_q}(\pi/2) e^{im_q\phi_q}. \quad (13)$$

As will be shown soon, instead of  $\phi_q$  only  $\phi_{qp} = \phi_q - \phi_p$  is needed. It is related to  $\omega_1$  and  $\omega_2$  by

$$\cos \phi_{qp} = \frac{1}{2qp}(\omega_2^2 - \omega_1^2 - M_2^2 + M_1^2), \quad (14)$$

with  $p$  and  $q$  from Eqs. (10) and (11), respectively. Thus we will take as independent variables besides the photon angles  $\Omega_\gamma = (\theta_\gamma, \phi_\gamma)$  and  $\phi_p$  the energies of the two mesons  $\omega_1$  and  $\omega_2$  instead of  $p$  and  $\phi_{qp}$  and obtain the following representation of the  $T$ -matrix element making the angular dependence explicit

$$T_{\nu\lambda\mu}(\phi_p, \omega_1, \omega_2, \Omega_\gamma) = e^{i(\lambda-\mu)\phi_\gamma} e^{-i\nu\phi_p} \sum_{JM} t_{\nu\lambda\mu}^{JM}(\omega_1, \omega_2) e^{-iM\phi_{\gamma p}} d_{M\lambda-\mu}^J(\theta_\gamma), \quad (15)$$

with the contribution of the final partial wave

$$\begin{aligned} t_{\nu\lambda\mu}^{JM}(\omega_1, \omega_2) &= t_{\nu\lambda\mu}^{JM}(\phi_{qp}) \\ &= \sum_{l_p j_p m_p L} \begin{pmatrix} l_p & \frac{1}{2} & j_p \\ 0 & \nu & -\nu \end{pmatrix} \begin{pmatrix} J & L & \frac{1}{2} \\ \mu - \lambda & \lambda & -\mu \end{pmatrix} d_{\nu m_p}^{j_p}(\pi/2) e^{i(M-m_p)\phi_{qp}} \mathcal{O}_M^{\lambda L J}(l_p j_p m_p), \end{aligned} \quad (16)$$

which shows the explicit dependence on  $\phi_{qp}$ . Furthermore, we have introduced for convenience the notation

$$\begin{aligned} \mathcal{O}_M^{\lambda L J}(l_p j_p m_p) &= \frac{(-1)^{1+J}}{2\sqrt{2\pi}} \sum_{l_q m_q} i^L (-1)^{l_p+j_p+l_q} \hat{l}_p \hat{j}_p \hat{l}_q \hat{L} d_{0 m_q}^{l_q}(\pi/2) \\ &\quad \times \begin{pmatrix} j_p & l_q & J \\ m_p & m_q & -M \end{pmatrix} \langle p q; ((l_p \frac{1}{2}) j_p l_q) J || \mathcal{O}^{\lambda L} || \frac{1}{2} \rangle. \end{aligned} \quad (17)$$

The following symmetry properties hold for the  $\mathcal{O}_M^{\lambda L J}(l_p j_p m_p)$

$$\mathcal{O}_M^{-\lambda L J}(l_p j_p m_p) = (-1)^{L+l_p+M-m_p} \mathcal{O}_M^{\lambda L J}(l_p j_p m_p), \quad (18)$$

$$\mathcal{O}_{-M}^{\lambda L J}(l_p j_p -m_p) = (-1)^{j_p+J} \mathcal{O}_M^{\lambda L J}(l_p j_p m_p), \quad (19)$$

where the first one is a consequence of parity conservation.

The symmetry relation of Eq. (9) leads to the following symmetry property of the amplitudes  $t_{\nu\lambda\mu}^{JM}$

$$t_{-\nu-\lambda-\mu}^{JM}(\phi_{qp}) = (-1)^{\nu+M} t_{\nu\lambda\mu}^{J-M}(-\phi_{qp}). \quad (20)$$

This means that for each  $J$  the number of independent amplitudes is  $4(2J+1)$ .

The complex functions  $t_{\nu\lambda\mu}^{JM}$ , depending on the meson energies  $\omega_1$  and  $\omega_2$  only, provide a complete description of the process in a manner analogous to the description of a single meson photoproduction in terms of multipoles. It is worth to point out, that in contrast to the binary reactions the partial amplitudes are functions of the c.m. energies of the final particles and, therefore, are to be determined for every point of the Dalitz plot.

## B. The differential cross section

For the unpolarized differential cross section one obtains with the  $T$ -matrix of Eq. (15)

$$\begin{aligned} \frac{d^4\sigma_0}{d\omega_1 d\omega_2 d\cos\theta_\gamma d\phi_{\gamma p}} &= c(W) \frac{1}{4} \sum_{\nu\lambda\mu} |T_{\nu\lambda\mu}|^2 \\ &= \sum_{jm} S_{jm}(\omega_1, \omega_2) Y_{jm}(\theta_\gamma, \phi_{\gamma p}) \end{aligned} \quad (21)$$

where we have defined

$$\begin{aligned} S_{jm} c(W)(\omega_1, \omega_2) &= \frac{\sqrt{\pi}}{2} c(W) \hat{j} \sum_{J' M' J M} (-1)^{-M'} \begin{pmatrix} J' & J & j \\ M' & -M & -m \end{pmatrix} \\ &\quad \times \sum_{\nu\lambda\mu} (-1)^{\lambda-\mu} \begin{pmatrix} J' & J & j \\ \lambda-\mu & \mu-\lambda & 0 \end{pmatrix} t_{\nu\lambda\mu}^{J' M'}(\omega_1, \omega_2)^* t_{\nu\lambda\mu}^{JM}(\omega_1, \omega_2), \end{aligned} \quad (22)$$

with

$$c(W) = \frac{M_N^2}{4(2\pi)^4(W^2 - M_N^2)} \quad (23)$$

as a kinematical factor. One should note that the differential cross section depends on the relative angle  $\phi_{\gamma p}$  only besides on  $\omega_1$ ,  $\omega_2$ , and  $\theta_\gamma$  as is immediately evident in the absence of polarization effects.

In terms of the electromagnetic multipole contributions one finds

$$\begin{aligned} S_{jm}(\omega_1, \omega_2) &= \frac{\sqrt{\pi}}{2} c(W) \hat{j} \sum_{J_p M_p} \hat{J}_p^2 d_{0M_p}^{J_p}(\pi/2) e^{i(m+M_p)\phi_{qp}} \\ &\times \sum_{l'_p j'_p m'_p l_p j_p m_p} (-1)^{j'_p - j_p - m_p} \begin{pmatrix} l'_p & l_p & J_p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j'_p & j_p & J_p \\ m'_p & -m_p & -M_p \end{pmatrix} \left\{ \begin{matrix} l'_p & l_p & J_p \\ j_p & j'_p & \frac{1}{2} \end{matrix} \right\} \\ &\times \sum_{J' M' J M L' L} (-1)^{J' + J + M' + L' + L'} \begin{pmatrix} J' & J & j \\ M' & -M & -m \end{pmatrix} \left\{ \begin{matrix} J' & J & j \\ L & L' & \frac{1}{2} \end{matrix} \right\} \\ &\times \sum_{\lambda} (-1)^{\lambda} \begin{pmatrix} L & L' & j \\ \lambda & -\lambda & 0 \end{pmatrix} \mathcal{O}_{M'}^{\lambda L' J'} (l'_p j'_p m'_p)^* \mathcal{O}_M^{\lambda L J} (l_p j_p m_p). \end{aligned} \quad (24)$$

If with respect to the fixed final state plane only the direction of the final nucleon is detected, one obtains a semi-inclusive differential cross section by integrating the expression in Eq. (21) over  $\omega_1$  and  $\omega_2$  (setting without loss of generality  $\phi_p = 0$ , which means that  $\phi_\gamma$  is measured relative to the direction of the nucleon momentum)

$$\begin{aligned} d\sigma_2/d\Omega_\gamma &= \int d\omega_1 d\omega_2 \frac{d^4\sigma_0}{d\omega_1 d\omega_2 d\Omega_\gamma} \\ &= \sum_{jm} \tilde{S}_{jm} Y_{jm}(\Omega_\gamma) \end{aligned} \quad (25)$$

as an expansion in terms of spherical harmonics in  $\Omega_\gamma$  with

$$\begin{aligned} \tilde{S}_{jm} &= \int d\omega_1 d\omega_2 S_{jm}(\omega_1, \omega_2) \\ &= \frac{\sqrt{\pi}}{2} c(W) \hat{j} \sum_{J' M' J M} (-1)^{-M'} \begin{pmatrix} J' & J & j \\ M' & -M & -m \end{pmatrix} \\ &\times \sum_{\nu\lambda\mu} (-1)^{\lambda-\mu} \begin{pmatrix} J' & J & j \\ \lambda-\mu & \mu-\lambda & 0 \end{pmatrix} \int d\omega_1 d\omega_2 t_{\nu\lambda\mu}^{J' M'}(\omega_1, \omega_2)^* t_{\nu\lambda\mu}^{J M}(\omega_1, \omega_2), \end{aligned} \quad (26)$$

or in terms of the multipoles

$$\begin{aligned} \tilde{S}_{jm} &= \frac{\sqrt{\pi}}{2} c(W) \hat{j} \sum_{J_p M_p} \hat{J}_p^2 d_{0M_p}^{J_p}(\pi/2) \sum_{l'_p j'_p m'_p l_p j_p m_p} (-1)^{j'_p - j_p - m_p} \begin{pmatrix} l'_p & l_p & J_p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j'_p & j_p & J_p \\ m'_p & -m_p & -M_p \end{pmatrix} \left\{ \begin{matrix} l'_p & l_p & J_p \\ j_p & j'_p & \frac{1}{2} \end{matrix} \right\} \\ &\times \sum_{J' M' J M L' L} (-1)^{J' + J + M' + L' + L'} \begin{pmatrix} J' & J & j \\ M' & -M & -m \end{pmatrix} \left\{ \begin{matrix} J' & J & j \\ L & L' & \frac{1}{2} \end{matrix} \right\} \\ &\times \sum_{\lambda} (-1)^{\lambda} \begin{pmatrix} L & L' & j \\ \lambda & -\lambda & 0 \end{pmatrix} \int d\omega_1 d\omega_2 e^{i(m+M_p)\phi_{qp}} \mathcal{O}_{M'}^{\lambda L' J'} (l'_p j'_p m'_p)^* \mathcal{O}_M^{\lambda L J} (l_p j_p m_p). \end{aligned} \quad (27)$$

Since  $d^2\sigma_0/d\Omega_\gamma$  is a real quantity, one has the property

$$\tilde{S}_{jm}^* = (-)^m \tilde{S}_{j-m}. \quad (28)$$

Furthermore, the cross section should be invariant under the simultaneous inversion of  $\vec{k}$  and  $\vec{p}$ , i.e. under the transformation  $\theta_{\gamma p} \rightarrow \pi - \theta_{\gamma p}$ . Thus one finds as additional symmetry property

$$\tilde{S}_{jm} = (-)^{j+m} \tilde{S}_{jm}, \quad (29)$$

from which the selection rule  $\tilde{S}_{jm} = 0$  for  $j + m = \text{odd}$  follows. This property can also be shown straightforwardly using Eq. (26) with the help of Eq. (19). For identical mesons, one finds from Eq. (20) an additional symmetry, namely

$$\tilde{S}_{j-m} = (-)^m \tilde{S}_{jm}, \quad (30)$$

which leads in conjunction with Eq. (28) to  $\Im m \tilde{S}_{jm} = 0$ .

It is more convenient to use instead of the differential cross section the corresponding normalized quantity

$$W(\Omega_\gamma) \equiv \frac{1}{\sigma_0} \frac{d^2\sigma_0}{d\Omega_\gamma} = \frac{1}{4\pi} + \sum_{jm, j \geq 1, j+m=\text{even}} \frac{\hat{j}}{\sqrt{4\pi}} W_{jm} Y_{jm}(\Omega_\gamma), \quad (31)$$

where the total cross section  $\sigma_0$  is given by

$$\begin{aligned} \sigma_0 &= 2\sqrt{\pi} \tilde{S}_{00} \\ &= \pi c(W) \int d\omega_1 d\omega_2 \sum_{\nu\lambda\mu JM} \frac{1}{2J+1} |t_{\nu\lambda\mu}^{JM}(\omega_1, \omega_2)|^2, \end{aligned} \quad (32)$$

and the expansion coefficients by

$$\begin{aligned} W_{jm} &= \frac{2\sqrt{\pi}}{\sigma_0 \hat{j}} \tilde{S}_{jm} \\ &= \frac{\pi}{\sigma_0} c(W) \int d\omega_1 d\omega_2 \\ &\quad \sum_{\nu\lambda\mu J' M' J M} (-1)^{\lambda+M+\mu} \begin{pmatrix} J' & J & j \\ M' & -M & 0 \end{pmatrix} \begin{pmatrix} J' & J & j \\ \lambda - \mu & \mu - \lambda & 0 \end{pmatrix} t_{\nu\lambda\mu}^{J' M}(\omega_1, \omega_2)^* t_{\nu\lambda\mu}^{J M}(\omega_1, \omega_2). \end{aligned} \quad (33)$$

Using the spherical harmonics expansion (31) should enable one to interpret the experimental results without resorting to a particular model. This expression is an analogue to the expansion of the single meson photoproduction cross section in terms of Legendre polynomials. The coefficients  $W_{jm}$  are hermitesch functionals of the partial amplitudes  $t_{\nu\lambda\mu}^{JM}$ . They obviously contain the whole information on the dynamics of the reaction with unpolarized particles and their values may in principle be extracted from the measurements and compared with model predictions. The selection rule  $W_{jm} = 0$  for  $j + m = \text{odd}$  may be used for a model independent partial wave analysis in the low energy region of the reaction, where usually only the first few waves contribute.

Otherwise an integration over the angles  $\theta_\gamma$  and  $\phi_\gamma$  gives the distribution of the events over the Dalitz plot

$$\frac{d^2\sigma}{d\omega_1 d\omega_2} = \pi c(W) \sum_{\nu\lambda\mu JM} \frac{1}{2J+1} |t_{\nu\lambda\mu}^{JM}(\omega_1, \omega_2)|^2. \quad (34)$$

Thus, as is well known, the partial waves of different  $J$  do not interfere in the Dalitz plot. In spite of its simplicity the expression in Eq. (34) can hardly be very useful in reconstructing even the modulae of the amplitudes  $t_{\nu\lambda\mu}^{JM}$ . Its use implies that one is able to establish a correspondence between variation of the amplitude as function of  $(\omega_1, \omega_2)$  and a specific value of the total angular momentum  $J$ . Obviously, for this purpose a detailed model is needed which relates  $J$  to particular decay channels. In this sense, using the moments  $W_{jm}$  should be more promising.

It is also clear that the information on the unpolarized differential cross section only is insufficient for a model independent determination of the amplitudes  $t_{\nu\lambda\mu}^{JM}$ . In the general case of photoproduction of two pseudoscalars eight independent complex functions are required to fix the spin structure of the amplitudes. Since the overall phase is always arbitrary, one has to measure 15 independent observables at each kinematical point. However, in certain cases, e.g., when the reaction is dominated by a single partial wave, using the moments  $W_{jm}$  enables one at least to draw a qualitative conclusion with respect to the partial wave structure. As an illustration, we consider in the next section the theoretically interesting case of  $\pi^0\pi^0$  and  $\pi^0\eta$  photoproduction on a proton.

### III. APPLICATION TO $\gamma p \rightarrow \pi^0\pi^0 p$ AND $\gamma p \rightarrow \pi^0\eta p$

The measured total cross section for  $\gamma p \rightarrow \pi^0\pi^0 p$  exhibits a rather steep rise in the energy region below the  $D_{13}(1520)$  resonance (see, e.g., [23]). At the same time, the existing models with a dominant contribution from

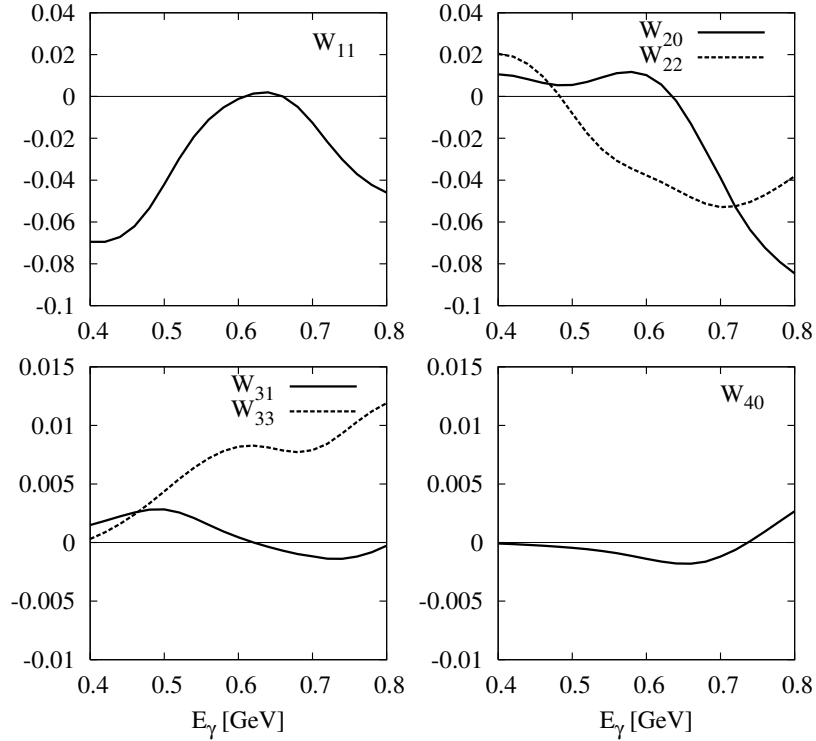


FIG. 2: The moments  $W_{jm}$  for  $\gamma p \rightarrow \pi^0 \pi^0 p$  as functions of the photon lab energy, normalized such that  $W_{00} = 1$ .

$D_{13}(1520)$  and a moderate role of the Roper resonance predict a cross section which increases rather slowly with increasing energy and is, therefore, far below the data. It is reasonable to assume that the almost linear energy dependence of the data indicates a contribution of a large fraction of  $s$  waves in the final state. The main mechanism providing the  $s$ -wave part in  $\pi\pi$  photoproduction is the  $\Delta$  Kroll-Ruderman term, appearing after minimal substitution of the electromagnetic interaction into the  $\pi N \Delta$  vertex. This term, however, vanishes in the neutral channel. The situation is similar to that in single  $\pi^0$  photoproduction at low energies. Here the Kroll-Ruderman does not enter the amplitude, thus leading to a visible suppression of the cross section for  $\gamma p \rightarrow \pi^0 p$  in comparison to the  $\pi^+$  or  $\pi^-$  case.

A possible large contribution of the Roper resonance  $P_{11}(1440)$  in the region  $E_\gamma = 500 - 600$  MeV as assumed in Ref. [11] seems to be excluded by more recent analyses. Furthermore, this assumption should be in disagreement with the experimental results of Ref. [24] for the helicity dependent total cross section  $\Delta\sigma = \sigma_{3/2} - \sigma_{1/2}$ . There it was found that in the energy region up to at least  $E_\gamma = 800$  MeV the  $3/2$  part dominates over the  $1/2$  part. This means, that the  $P_{11}$  wave, which contributes only to  $\sigma_{1/2}$ , should be overwhelmed by the waves with higher spins.

Thus the question concerning the partial wave structure of the amplitude for  $\gamma N \rightarrow \pi^0 \pi^0 N$  is still open. In order to reveal in this case the mechanism responsible for an unusually large fraction of the  $s$  wave part in the  $\pi^0 \pi^0$  amplitude, it is useful to analyse the moments  $W_{jm}$  throughout the energy range from threshold up to the  $D_{13}(1520)$  peak. In order to keep the number of parameters limited, one can use only the lowest partial waves. Their choice is inspired by the previous isobar model analyses of Refs. [10, 12, 14] showing that only waves with  $J \leq 5/2$  are important below  $E_\gamma = 1$  GeV.

As an example we show in Fig. 2 the variation of  $W_{jm}$  for  $j \leq 3$  as predicted by the  $\pi\pi$  model of Ref. [14]. The model [14] is based on a traditional phenomenological Lagrangean approach with Born and resonance amplitudes calculated on the tree level. The interaction within the  $\pi N$  and  $\pi\pi$  pairs is effectively taken into account via  $\Delta$ ,  $\rho$  and  $\sigma$ . The  $\pi\pi N$  state is then produced through intermediate formation of  $\pi\Delta$ ,  $\rho N$  and  $\sigma N$  channels. The contributions from the resonances are parametrized in the usual way in terms of a Breit-Wigner ansatz with energy-dependent widths. For the parameters of the model, i.e. masses, partial widths and electromagnetic couplings of resonances, the corresponding average values from the compilation of the Particle Data Group were used.

In case of  $\pi^0 \pi^0$  production due to the identity of the two mesons we have an additional symmetry relation

$$W(\theta_\gamma, \phi_\gamma) = W(\theta_\gamma, 2\pi - \phi_\gamma), \quad (35)$$

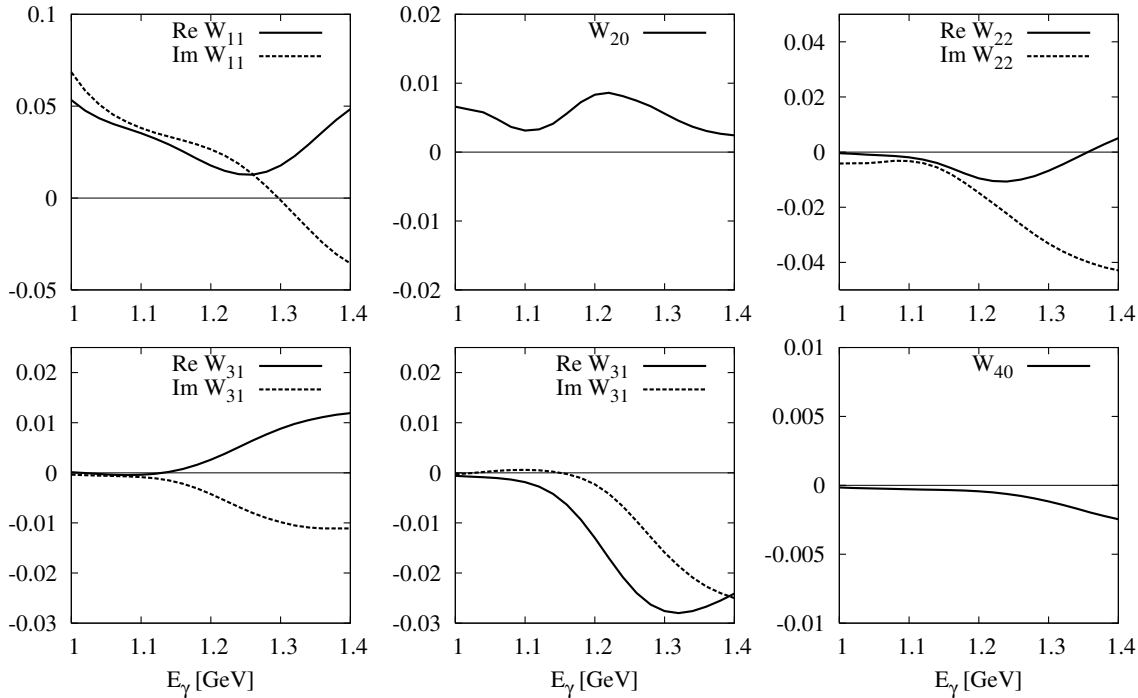


FIG. 3: Same as in Fig. 2 for  $\gamma p \rightarrow \pi^0 \eta p$ . The dashed lines represent the imaginary parts.

which is a consequence of the symmetry property in Eq. (30). The moments for  $j = 3, 4$  are small as are those for higher values of  $j$  which are not shown. In the region  $E = 650 - 800$  MeV the moments  $W_{11}$  and  $W_{20}$  exhibit a crucial energy dependence due to the  $D_{13}(1520)$  resonance, dominating the reaction  $\gamma p \rightarrow \pi^0 \pi^0 p$  at this energy. Large values of the moments with  $j$  odd indicate the presence of waves with opposite parities. In particular, the structure in  $W_{11}$  is due to an interference between the wave  $J^P = 3/2^-$  dominated by  $D_{13}(1520)$  and the waves  $J^P = 1/2^+$  and  $3/2^+$ . The latter are saturated, apart from the Roper resonance, by the Born terms. The contribution of  $W_{11}$  becomes minimal in magnitude in the region around  $E_\gamma = 650$  MeV, where the real part of the  $D_{13}(1520)$  propagator vanishes, and it interferes weakly with the predominantly real Born amplitudes. Thus, if our notion about the  $\pi^0 \pi^0$  photoproduction mechanism is correct we expect a rather small value of the moments  $W_{20}$  and  $W_{22}$  and a relatively large value of  $W_{11}$  in the region below the  $D_{13}(1520)$  peak.

In this respect we would like to note that according to the fit in Ref. [16] there must be a large contribution of the resonance  $D_{33}(1700)$  to the channel  $\pi^0 \pi^0 p$  in a wide energy range from the lowest energies up to  $E_\gamma = 1.4$  GeV. In particular, inclusion of this resonance into the amplitude explains both the steep rise of the total cross section below  $E_\gamma = 700$  MeV and the second peak observed at  $E_\gamma = 1.1$  GeV. If the resonance  $D_{33}(1700)$  is indeed so important in the  $\pi^0 \pi^0$  channel, it should increase the values of  $W_{20}$  and  $W_{22}$ . All in all, a measurement of these moments will help us to understand the role of  $d$ -wave resonances with  $J = 3/2$  in  $\pi^0 \pi^0$  photoproduction.

As for  $\pi^0 \eta$  photoproduction, the partial wave structure of the corresponding amplitude was investigated in detail in Refs. [2, 15, 17]. There it was shown that the  $J^P = 3/2^-$  wave, containing  $D_{33}(1700)$  and probably  $D_{33}(1940)$ , apparently dominates the reaction in a wide region from threshold to about  $E_\gamma = 1.7$  GeV. Other waves, primarily  $1/2^+$  and  $5/2^+$ , manifest themselves in angular distributions of the final particles mostly via interference with the dominant  $3/2^-$  wave.

In Fig. 3 we present the energy dependence of the expansion coefficients for  $\gamma p \rightarrow \pi^0 \eta p$  obtained using the isobar model of Ref. [17]. Here the relation (30) does not hold, so that the moments  $W_{jm}$  with  $m \neq 0$  have nonvanishing imaginary parts (dashed lines in Fig. 3). The calculation follows the same line as for the  $\pi^0 \pi^0$  case. Namely, the final  $\pi^0 \eta N$  state results from the two step decay of baryon resonances via the intermediate quasi-two-body channels  $\eta \Delta$  and  $\pi^0 S_{11}(1535)$ . The parameters of the model were fitted to the angular distributions of the final particles measured in Ref. [6]. The fitting procedure is described in [17] and the reader is referred to this work for more details.

Firstly, as one can see in Fig. 3 in spite of the mentioned dominance of the  $3/2^-$  wave, the values of  $W_{20}$  and  $W_{22}$  are small. This is because of the closeness of the  $3/2$  and  $1/2$  helicity couplings of the resonance  $D_{13}(1700)$  (see,



e.g., the discussion in Ref. [17]). As a result, the hermitian forms of  $t_{\nu\lambda\mu}^{3/2M}$  entering  $W_{20}$  and  $W_{22}$  according to (33) almost cancel each other. At the same time, we obtain a rather large value of the coefficient  $W_{11}$ , mainly determined by the interference between the resonances  $D_{33}(1700)$  and  $P_{31}(1750)$ . According to these results we may expect that the data for  $\pi^0\eta$  will show relatively small values of all moments except for  $W_{11}$ . If this prediction is not confirmed by measurements one has to critically review the existing conceptions about the dynamics of  $\pi^0\eta$  photoproduction, based on the results from Refs. [2, 7, 9, 17, 18].

#### IV. CONCLUSION

Practical methods for the analysis of the partial wave structure of reactions with three particles in the final state are obviously needed for the study of the dynamical features of two-meson photoproduction. The formalism used in the present paper specifies the final  $\pi\pi N$  states by means of two c.m. energies and two angles, determining the orientation of the final state momentum triangle (final state three-particle plane) with respect to the beam axis. The partial wave decomposition may then be performed via a transition from the continuum variables (angles) to the set of discrete variables  $JM$  being the total angular momentum  $J$  and its projection  $M$  on the normal to the three-particle plane. The corresponding partial wave amplitudes  $t_{\nu\lambda\mu}^{JM}$  contain the whole information on the production dynamics. We would like to stress the fact that this method does not involve a decomposition with respect to the angular momenta of the final two-body subsystems and is in principle free from any assumptions about the production mechanism.

In the present paper we have considered only the unpolarized differential cross section. Although this quantity does not allow a unique determination of the amplitudes  $t_{\nu\lambda\mu}^{JM}$ , the information on the angular distribution of the participating particles can serve to place restrictions on contributions of states with definite angular momentum and parity. This in turn is crucial for our understanding of the resonance content of the reaction. For this purpose the differential cross section has been expanded in terms of spherical harmonics with coefficients or moments  $W_{jm}$  in a manner similar to the representation of the binary cross section in terms of Legendre polynomials.

#### Acknowledgment

This work was supported by the Deutsche Forschungsgemeinschaft (SFB 443, SFB 1044) and by the Russian Federal programm “Kadry” (contract 16.740.11.0469). Furthermore, A. Fix thanks the Institut für Kernphysik in Mainz for their hospitality.

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